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THE OSCILLATOR CORRESPONDENCE OF ORBITAL INTEGRALS, FOR PAIRS OF TYPE ONE IN THE STABLE RANGE

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1. Introduction. Let G, $G' \subseteq Sp(W)$ be a reductive dual pair of type I; see [H2]. Thus, there is a division algebra $\mathbf{D} = (\mathbf{R}, \mathbf{C}, \mathbf{H})$ with an involution over $\mathbf{R}$, two finite-dimensional vector spaces over $\mathbf{D}$, V and V' equipped with nondegenerate forms $(\,,\,)$ and $(\,,\,)'$, respectively—one hermitian and the other skew-hermitian. The groups G, G' are the isometry groups of the forms $(\,,\,)$, $(\,,\,)'$, respectively. Let W denote the vector space $W = \mathrm{Hom}(V', V)$. A symplectic form on W is defined by

$$\langle w, w' \rangle = \mathrm{tr}_{\mathbf{D}/\mathbf{R}}(ww'^*) \qquad (w, w' \in W), \tag{1.1}$$

where the map $\mathrm{Hom}(V', V) \ni w \to w^* \in \mathrm{Hom}(V, V')$ is defined by

$$(w(v'), v) = (v', w^*(v))' \qquad (w \in W, v \in V, v' \in V'). \tag{1.2}$$

The groups G and G' act on W via postmultiplication and premultiplication by the inverse, respectively. These actions embed G and G' into the symplectic group $Sp(W)$.

Let $\widetilde{Sp}$ denote the metaplectic group, and let $\tilde{G}$, $\tilde{G}'$ be the preimages of G, G' under the covering map $\widetilde{Sp} \to Sp$. The duality theorem of Howe [H3] states that there is a bijection $\Pi \leftrightarrow \Pi'$ between certain irreducible admissible representations of $\tilde{G}$ and $\tilde{G}'$.

Recall the unnormalized moment maps

$$\tau_{\mathfrak{g}}\colon W \ni w \to ww^* \in \mathfrak{g}, \qquad \tau_{\mathfrak{g}'}\colon W \ni w \to w^*w \in \mathfrak{g}'. \tag{1.3}$$

In the early 1980s, Howe conjectured that the wave-front sets of Π and Π' are related to the geometry of moment maps in some nice way.

Conjecture (Howe). *For a generic pair (Π, Π') occurring in Howe's correspondence,*

$$WF(\Pi') = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(WF(\Pi))). \tag{1.4}$$

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The equality (1.4) was proven in [P. 7.10] under a very strong assumption that the pair G, G' is in the stable range, with G the smaller member, and that the representation Π is unitary and finite-dimensional.

In this paper, we propose another approach towards a proof of the conjecture (1.4). Let Θ_Π denote the (distribution) character of Π, and let u_Π be the lowest term in the asymptotic expansion of Θ_Π at the identity, as defined in [BV]. Similarly, we have $\Theta_{\Pi'}$ and $u_{\Pi'}$. We would like to state a conjecture relating u_Π and $u_{\Pi'}$ to the geometry of moment maps. Before we do it, we need some preparation.

In [H1], Howe has deduced from Witt's theorem the following.

THEOREM 1.5. *There is an open dense $G \cdot G'$-invariant subset $W^{\max} \subseteq W$ such that for every orbit $\mathcal{O} \subseteq \tau_{\mathfrak{g}}(W^{\max})$, the set $\mathcal{O}' = \tau_{\mathfrak{g}'}(W^{\max} \cap \tau_{\mathfrak{g}}^{-1}(\mathcal{O}))$ is a single G' orbit.*

The set $W^{\max}$ is not unique, of course. In this paper, we assume that the pair G, G' is in the stable range, with G the smaller member. This means that V' has an isotropic subspace of dimension greater than or equal to the dimension of V. We shall prove the following.

THEOREM 1.6. *There is an affine section $\sigma_{\mathfrak{g}}\colon \mathfrak{g} \to W$ to the map $\tau_{\mathfrak{g}}$, a function $m(z, g')$ on $\mathfrak{g} \times G'$ and a (singular) measure μ on G' such that*

$$\text{(a)}\quad \int_W \phi(w)\, dw = \int_{\mathfrak{g}} \int_{G'} \int_G \phi(g \cdot \sigma_{\mathfrak{g}}(z) \cdot g'^{-1})\, dg\, m(z, g')\, d\mu(g')\, dz \qquad (\phi \in C_c(W)),$$

and if $\psi\colon \mathfrak{g}' \to \mathbf{C}$ is a continuous and rapidly decreasing function, then so is

$$\text{(b)}\qquad \mathcal{A}\psi(z) = \int_{G'} \psi \circ \tau_{\mathfrak{g}'}(\sigma_{\mathfrak{g}}(z) \cdot g'^{-1}) m(z, g')\, d\mu(g') \qquad (z \in \mathfrak{g}).$$

Theorem 1.5 holds for the set $W^{\max} = \{g \cdot \sigma_{\mathfrak{g}}(z) \cdot g', g \in G, z \in \mathfrak{g}, g' \in G'\}$. Furthermore, if ψ is smooth and $\operatorname{supp} \psi \cap \tau_{\mathfrak{g}'}(W^{\max})$ is compact, then $\mathcal{A}\psi \in S(\mathfrak{g})$, the Schwartz space of $\mathfrak{g}$.

For an explicit formulation, see (2.24), (2.25), (3.15), and (3.16). The case $G = Sp_{p,q}$, $G' = O^*{}_{2n}$ can be treated similarly, and it is left to the reader.

Let $\mu_{\mathcal{O}}$ be the canonical invariant measure on a G-orbit $\mathcal{O} \subseteq \mathfrak{g}$. Then by [RR], $\mu_{\mathcal{O}}$ can be integrated against any rapidly decreasing function on $\mathfrak{g}$. Thus, in view of the above, we may define a measure $\mathcal{A}^*\mu_{\mathcal{O}}$ on $\mathfrak{g}'$ by

$$\mathcal{A}^*\mu_{\mathcal{O}}(\psi) = \mu_{\mathcal{O}}(\mathcal{A}\psi),$$

where ψ is a rapidly decreasing function on $\mathfrak{g}'$. It will be clear from Theorem 1.6 and from the following construction that the measure $\mathcal{A}^*\mu_{\mathcal{O}}$ is invariant and is supported on the closure of $\mathcal{O}'$.

THEOREM 1.7. *With the above notation, we have* $\mathscr{A}^*\mu_{\mathcal{O}} = \mathrm{const}\cdot\mu_{\mathcal{O}'}$, *where* $\mathrm{const} > 0$ *and* $\mathcal{O}' \subseteq \mathfrak{g}'$ *is the* G'*-orbit corresponding to* $\mathcal{O}$ *via the Howe-Witt theorem (Theorem 1.5).*

Let $\kappa(\ ,\ )$ denote the Killing form on $\mathfrak{g}$ and define a Fourier transform by

$$\hat{\psi}(x) = \int_{\mathfrak{g}} \psi(y)e^{i\kappa(x,y)}\,dy \qquad (\psi \in S(\mathfrak{g}),\ x \in \mathfrak{g}).$$

Let $\hat{\mu}_{\mathcal{O}} \in S^*(\mathfrak{g})$ denote the Fourier transform of $\mu_{\mathcal{O}}$. By Harish-Chandra, this distribution coincides with a function $\hat{\mu}_{\mathcal{O}}(z)$, $z \in \mathfrak{g}$; see [W, 8.3.5]. Similarly, we have $\hat{\mu}_{\mathcal{O}'}(z')$, $z' \in \mathfrak{g}'$. By combining (1.6) and (1.7) with the fact that $\hat{\mu}_{\mathcal{O}}$ is absolutely integrable against any Schwartz function, we deduce the following theorem.

THEOREM 1.8. *There is a constant* $\mathrm{const} > 0$ *such that for* $\psi \in S(\mathfrak{g})$ *with* $\operatorname{supp}\hat{\psi} \cap \tau_{\mathfrak{g}'}(W^{\max})$ *compact*

$$\hat{\mu}_{\mathcal{O}'}(\psi) = \int_{\mathfrak{g}'} \psi(z')\hat{\mu}_{\mathcal{O}'}(z')\,dz' = \mathrm{const}\int_{\mathfrak{g}} (\mathscr{A}(\hat{\psi}))\hat{\ }(z)\hat{\mu}_{\mathcal{O}}(z)\,dz,$$

where the integrals are absolutely convergent.

Now we can state our conjecture.

CONJECTURE. *There is a constant* $\mathrm{const} > 0$, *depending only on normalization of the Lebesgue measure on* $\mathfrak{g}'$, *such that*

$$\hat{u}_{\Pi'} = \mathscr{A}^*\hat{\bar{u}}_{\Pi}, \tag{1.9}$$

where $\bar{u}_{\Pi}$ *stands for the complex conjugate of the function* u_{Π}.

Thanks to [R] and Theorems 1.5, 1.7, and 1.8, equation (1.9) would imply (1.4). Recently, we have proved that (1.9) holds in the "deep stable range" (see [DP]), where we can compute the distribution character of Π' from that of Π. Although proving the conjecture in the general situation is at present more a matter of hard work than insight, a proof has not yet been written down.

2. The case when $(\ ,\ )'$ is hermitian. In this section, $\mathbf{D}$ is equipped with an involution $\mathbf{D} \ni a \to \bar{a} \in \mathbf{D}$, which is trivial only if $\mathbf{D} = \mathbf{R}$. Let $M_{m,n}(\mathbf{D})$ denote the set of matrices with m rows and n columns and with entries from $\mathbf{D}$. Let $M_n(\mathbf{D}) = M_{n,n}(\mathbf{D})$ and let $\mathbf{D}^n = M_{n,1}(\mathbf{D})$. We view $\mathbf{D}^n$ as a left vector space over $\mathbf{D}$ by the following formula

$$av = v\cdot\bar{a} \qquad (a \in \mathbf{D},\ v \in \mathbf{D}^n).$$

Each matrix $F \in M_n(\mathbf{D})$ acts on $\mathbf{D}^n$ via left multiplication. Thus, $M_n(\mathbf{D})$ may be identified with $\mathrm{End}_{\mathbf{D}}(\mathbf{D}^n)$. Since $\mathbf{R} \subseteq \mathbf{D}$, $\mathbf{D}^n$ may be viewed as a real vector space, and we have an obvious inclusion $\mathrm{End}_{\mathbf{D}}(\mathbf{D}^n) \subseteq \mathrm{End}_{\mathbf{R}}(\mathbf{D}^n)$. For $F \in M_n(\mathbf{D})$, let $\det_{\mathbf{R}}(F)$ denote the determinant of F viewed as an element of $\mathrm{End}_{\mathbf{R}}(\mathbf{D}^n)$.

For two positive integers $d' \geqslant 2d$, set $V = \mathbf{D}^d$ and $V' = \mathbf{D}^{d'}$. Fix a matrix $F \in M_d(\mathbf{D})$ such that $F = -\bar{F}^t$ and $|\det_{\mathbf{R}}(F)| = 1$. Let

$$F' = \begin{bmatrix} 0 & 0 & I_d \\ 0 & F'' & 0 \\ I_d & 0 & 0 \end{bmatrix}, \quad F'' = \begin{pmatrix} I_{p'} & 0 \\ 0 & -I_{q'} \end{pmatrix},$$

$$2d + p' + q' = d', \quad p = p' + d, \quad q = q' + d.$$

Set

$$(u, v) = \bar{u}^t F v, \qquad (u', v')' = \bar{u'}^t F' v' \quad (u, v \in V, u', v' \in V'). \tag{2.1}$$

Then $(\,,\,)$ is a nondegenerate skew-hermitian form on V and $(\,,\,)'$ is a nondegenerate hermitian form on V' of signature p, q. The corresponding isometry groups and Lie algebras can be represented in terms of matrices as follows:

$$G = \{g \in M_d(\mathbf{D}); \bar{g}^t F g = F\}, \qquad \mathfrak{g} = \{z \in M_d(\mathbf{D}); \bar{z}^t F + F z = 0\},$$

$$G' = \{g \in M_{d'}(\mathbf{D}); \bar{g}^t F' g = F'\}, \qquad \mathfrak{g}' = \{z \in M_{d'}(\mathbf{D}); \bar{z}^t F' + F' z = 0\}.$$

Let $W = M_{d,d'}(\mathbf{D})$. This is a symplectic space over $\mathbf{R}$ with the symplectic form defined in terms of the forms (2.1) as in (1.1). Let $L' = \{g \in G'; \bar{g}^t g = I_{d'}\}$. This is a maximal compact subgroup of G'. The centralizer of L' in $Sp(W)$ is isomorphic to $L = G \times G$. Let $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{g}$. Then we have the moment maps $\tau_{\mathfrak{g}}: W \to \mathfrak{g}$, $\tau_{\mathfrak{g}'}: W \to \mathfrak{g}'$ and $\tau_{\mathfrak{l}}: W \to \mathfrak{l}$ given explicitly by

$$\begin{aligned} &\tau_{\mathfrak{g}}(w) = wF'\bar{w}^t F, \qquad \tau_{\mathfrak{l}}(w) = ((w\bar{w}^t + wF'\bar{w}^t)F, -(w\bar{w}^t - wF'\bar{w}^t)F), \\ &\tau_{\mathfrak{g}'}(w) = F'\bar{w}^t F w \qquad (w \in W). \end{aligned} \tag{2.2}$$

(These maps $\tau_{\mathfrak{g}}, \tau_{\mathfrak{g}'}, \tau_{\mathfrak{l}}$ are essentially determined by the fact that they are constant on the G', G, L' orbits in W, respectively.) We shall view W as a direct sum

$$W = M_d(\mathbf{D}) \oplus M_{d,d'-2d}(\mathbf{D}) \oplus M_d(\mathbf{D}), \tag{2.3}$$

where each $w \in W$ is written as $w = (a, b, c)$, $a \in M_d(\mathbf{D})$, $b \in M_{d,d'-2d}(\mathbf{D})$, $c \in M_d(\mathbf{D})$. In terms of (2.3), define an affine map $\sigma_{\mathfrak{g}}: \mathfrak{g} \to W$

$$\sigma_{\mathfrak{g}}(z) = \left(\frac{1}{2}z, 0, -F^{-1}\right) \qquad (z \in \mathfrak{g}). \tag{2.4}$$

We shall see in (2.7) that this is a section to the map $\tau_{\mathfrak{g}}$.

We shall identify the general linear group $GL(V)$ with a subgroup of G' by the following injective group homomorphism

$$GL(V) \ni g \to \begin{bmatrix} g & 0 & 0 \\ 0 & I_{d'-2d} & 0 \\ 0 & 0 & (\bar{g}^t)^{-1} \end{bmatrix} \in G'. \tag{2.5}$$

Then for $g \in GL(V)$ and $z \in \mathfrak{g}$

$$g(\sigma_{\mathfrak{g}}(z)) = \sigma_{\mathfrak{g}}(z) \cdot g^{-1} = \left(\frac{1}{2} z g^{-1}, 0, -F^{-1}\bar{g}^t\right) \tag{2.6}$$

and

$$\begin{aligned} \tau_{\mathfrak{g}}(g(\sigma_{\mathfrak{g}}(z))) &= z \\ \tau_{\mathfrak{l}}(g(\sigma_{\mathfrak{g}}(z))) &= \left(\left(\frac{1}{4} z g^{-1}(\bar{g}^t)^{-1}\bar{z}^t + (\overline{F^{-1}})^t \bar{g}^t g F^{-1} + z F^{-1}\right) F,\right. \\ &\qquad \left. -\left(\frac{1}{4} z g^{-1}(\bar{g}^t)^{-1}\bar{z}^t + (\overline{F^{-1}})^t \bar{g}^t g F^{-1} - z F^{-1}\right) F\right) \end{aligned}$$

$$\tau_{\mathfrak{g}'}(g(\sigma_{\mathfrak{g}}(z))) = \begin{bmatrix} \frac{1}{2} g z g^{-1} & 0 & -gF^{-1}\bar{g}^t \\ 0 & 0 & 0 \\ \frac{1}{4}(\bar{g}^t)^{-1}\bar{z}^t F z g^{-1} & 0 & -\frac{1}{2}(\bar{g}^t)^{-1}\bar{z}^t\bar{g}^t \end{bmatrix}. \tag{2.7}$$

Before proceeding any further, we make the following observation. With the notation of (2.7), let $S = zF^{-1}$ and let $T = (\overline{F^{-1}})^t \bar{g}^t g F^{-1}$. Then

$$\tau_{\mathfrak{l}}(g(\sigma_{\mathfrak{g}}(z))) = \left(\left(\frac{1}{4} S T^{-1} \bar{S}^t + T + S\right) F, -\left(\frac{1}{4} S T^{-1} \bar{S}^t + T - S\right) F\right). \tag{2.8}$$

Let $\mathscr{H} = \{S \in M_d(\mathbf{D}); S = \bar{S}^t\}$ be the space of hermitian matrices of size d. Let $\mathscr{H}^+ = \{S \in \mathscr{H}; S > 0\}$ be the subset of positive definite matrices. For $S \in \mathscr{H}$, let $\mathscr{H}_S^+ = \{T \in \mathscr{H}^+; T > (1/4) S T^{-1} S\}$, and let $\mathscr{H}_{\pm S}^+ = \{P \in \mathscr{H}^+; P \pm S > 0\}$.

LEMMA 2.9. *Fix $S \in \mathscr{H}$. Then the map*

$$\mathscr{H}_S^+ \ni T \to \frac{1}{4} S T^{-1} S + T \in \mathscr{H}_{\pm S}^+$$

is a bijection.

Proof. Suppose first that $d = 1$ and $\mathbf{D} = \mathbf{R}$. Then the above statement means that for any $s \in \mathbf{R}$, the map

$$\left(\frac{1}{2}|s|, +\infty\right) \ni t \to \frac{1}{4}s^2t^{-1} + t \in (|s|, +\infty) \tag{2.10}$$

is a bijection. This is elementary.

Notice that for $g \in GL(V)$

$$g\left(\frac{1}{4}ST^{-1}S + T\right)\bar{g}^t = \frac{1}{4}(gS\bar{g}^t)(gT\bar{g}^t)^{-1}(gS\bar{g}^t) + gT\bar{g}^t.$$

Hence, by the spectral theorem for hermitian matrices, we may assume that

$$\frac{1}{2}S = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}.$$

The stabilizer of $(1/2)S$ in $GL(V)$ (under the above action) consists of matrices of the form

$$g = \begin{pmatrix} h & B \\ 0 & C \end{pmatrix}, \qquad hE\bar{h}^t = E, \qquad \det_{\mathbf{R}}(C) \neq 0. \tag{2.11}$$

Suppose we know that the lemma holds if $d = r + s$ (i.e., *if* $(1/2)S = E$). Let $d > r + s$. We shall write $T \in \mathscr{H}^+$ in a block form

$$T = \begin{pmatrix} T_1 & T_2 \\ \bar{T}_2^t & T_3 \end{pmatrix}$$

as in (2.11). Notice that $T_3 > 0$. Take $B = -hT_2T_3^{-1}$ in (2.11). Then

$$gT\bar{g}^t = \begin{pmatrix} h(T_1 - T_2T_3^{-1}\bar{T}_2{}^t)\bar{h}^t & 0 \\ 0 & CT_3\bar{C}^t \end{pmatrix}. \tag{2.12}$$

Thus, elements of $\mathscr{H}^+$ are diagonalizable via the action of the stabilizer of $(1/2)S$. (This shall be verified shortly for the case $(1/2)S = E$.) Hence, by (2.10) the map (2.9) is surjective.

Suppose $T, T' \in \mathscr{H}_S^+$ and

$$\frac{1}{4}ST^{-1}S + T = \frac{1}{4}ST'^{-1}S + T'. \tag{2.13}$$

Write T' in a block form as in (2.11):

$$T' = \begin{pmatrix} T_1' & T_2' \\ \overline{T}_2'^t & T_3' \end{pmatrix}.$$

Then (2.13) implies that $T_2 = T_2'$ and $T_3 = T_3'$. Thus, the same g as in (2.12) gives

$$gT'\overline{g}^t = \begin{pmatrix} h(T_1' - T_2'T_3'^{-1}\overline{T}_2'^t)\overline{h}^t & 0 \\ 0 & CT_3'\overline{C}^t \end{pmatrix}. \tag{2.14}$$

By combining (2.12–2.14), we get

$$\begin{aligned} &\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h(T_1 - T_2T_3^{-1}\overline{T}_2^t)\overline{h}^t & 0 \\ 0 & CT_3\overline{C}^t \end{pmatrix}^{-1} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} h(T_1 - T_2T_3^{-1}\overline{T}_2^t)\overline{h}^t & 0 \\ 0 & CT_3\overline{C}^t \end{pmatrix} \\ &= \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h(T_1' - T_2T_3^{-1}\overline{T}_2^t)\overline{h}^t & 0 \\ 0 & CT_3\overline{C}^t \end{pmatrix}^{-1} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} h(T_1' - T_2T_3^{-1}\overline{T}_2^t)\overline{h}^t & 0 \\ 0 & CT_3\overline{C}^t \end{pmatrix}. \end{aligned}$$

Hence, by taking the terms in the upper-left corner, we see that

$$\begin{aligned} &E((T_1 - T_2T_3^{-1}\overline{T}_2^t))^{-1}E + (T_1 - T_2T_3^{-1}\overline{T}_2^t) \\ &\quad = E((T_1' - T_2T_3^{-1}\overline{T}_2^t))^{-1}E + (T_1' - T_2T_3^{-1}\overline{T}_2^t). \end{aligned}$$

Therefore (by our assumption that the lemma holds for $d = r + s$),

$$T_1 - T_2T_3^{-1}\overline{T}_2^t = T_1' - T_2T_3^{-1}\overline{T}_2^t.$$

Thus, $T_1 = T_1'$ and consequently $T = T'$. Hence, the map (2.9) is injective.

From now on, we assume $d = r + s$. Let H denote the stabilizer of $(1/2)S = E$ in $GL(V)$. Let $\overline{B}^+ = \{b = \mathrm{diag}(b_1, b_2, \ldots, b_d);\ b_1 \geqslant b_2 \geqslant \cdots \geqslant b_r > 0,\ b_{r+1} \geqslant b_{r+2} \geqslant \cdots \geqslant b_d > 0\}$. Let $U = \{g \in GL(V);\ g\overline{g}^t = I_d\}$. By a well-known structure theorem for symmetric spaces [S, 7.1.3], the map

$$H \times \overline{B}^+ \times U \ni (h, b, u) \to hbu \in GL(V) \tag{2.15}$$

is surjective with the fiber $\{(hl, b, l^{-1}u);\ l \in \text{centralizer of } b \text{ in } H \cap U\}$. In particular, we see that the action of H on $\mathscr{H}^+$ is diagonalizable. Hence, (2.10) implies that the map (2.9) is surjective.

It remains to prove the injectivity. Suppose $T = hb^2\bar{h}^t$ and $T' = h'b'^2\bar{h}'^t \in \mathscr{H}_S^+$ satisfy (2.13). Then

$$E(hb^2\bar{h}^t)^{-1}E + hb^2\bar{h}^t = E(h'b'^2\bar{h}'^t)^{-1}E + h'b'^2\bar{h}'^t,$$

so

$$E(b^2)^{-1}E + b^2 = (h^{-1}h')(E(b'^2)^{-1}E + b'^2)(\overline{h^{-1}h'})^t.$$

But E commutes with b and $E^2 = I_d$. Hence,

$$b^{-2} + b^2 = (h^{-1}h')(b'^{-2} + b'^2)(\overline{h^{-1}h'})^t. \tag{2.16}$$

Moreover, the condition $T, T' \in \mathscr{H}_S^+$ implies that $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_r > 1$ and $b_{r+1} \geqslant b_{r+2} \geqslant \cdots \geqslant b_d > 1$. Notice that if $y \geqslant x \geqslant 1$, then $y + y^{-1} \geqslant x + x^{-1}$. Hence, $b^{-2} + b^2 \in \bar{B}^+$ and $b'^{-2} + b'^2 \in \bar{B}^+$. Therefore, (2.15) and (2.16) imply that $b^{-2} + b^2 = b'^{-2} + b'^2$ and $h^{-1}h' = l$ for some l in the centralizer of $b^{-2} + b^2$ in $H \cap U$. Notice that b can be written in terms of $c = b^{-2} + b^2$

$$b = \sqrt{\frac{c + \sqrt{c^2 - 4}}{2}}\ .$$

Hence, l commutes with b. Therefore, $T' = h'b'^2\bar{h}'^t = hlb^2l^{-1}\bar{h}^t = hb^2\bar{h}^t = T$. $\square$

COROLLARY 2.17. *Let* $(\mathscr{H} \times \mathscr{H}^+)^+ = \{(S, T) \in \mathscr{H} \times \mathscr{H}^+;\ T > (1/4)ST^{-1}S\}$. *Then the map*

$$(\mathscr{H} \times \mathscr{H}^+)^+ \ni (S, T) \to \left(\frac{1}{4}ST^{-1}S + T + S, \frac{1}{4}ST^{-1}S + T - S\right) \in \mathscr{H}^+ \times \mathscr{H}^+$$

is a bijection.

Proof. Given $P, P' \in \mathscr{H}^+$, we want to show that there is a unique $(S, T) \in (\mathscr{H} \times \mathscr{H}^+)^+$ such that

$$\frac{1}{4}ST^{-1}S + T + S = P$$

$$\frac{1}{4}ST^{-1}S + T - S = P'.$$

Clearly, $S = (1/2)(P - P')$. Notice that $P + P' \pm (P - P') > 0$. Thus, $(1/2)(P + P') \in \mathscr{H}^+_{\pm S}$. Hence, by (2.9) there is a unique $T \in \mathscr{H}_S^+$ such that $(1/4)ST^{-1}S + T = (1/2)(P + P')$. $\square$

Define a representation ρ of $GL(V)$ on the real vector space $\mathscr{H}$ by

$$\rho(g)S = gS\bar{g}^t \qquad (g \in GL(V), S \in \mathscr{H}). \tag{2.18}$$

COROLLARY 2.19. *Let dP denote a Lebesgue measure on $\mathscr{H}$. Then there is* const > 0 *such that for a test function* ψ

$$\int_{\mathscr{H}^+}\int_{\mathscr{H}^+} \psi(P, P')\, dP'\, dP = \text{const} \int_{(\mathscr{H} \times \mathscr{H}^+)^+} \psi\left(\frac{1}{4}ST^{-1}S + T + S, \right.$$
$$\left.\frac{1}{4}ST^{-1}S + T - S\right)\left|\det_{\mathbf{R}}\left(1 - \rho\left(\frac{1}{2}ST^{-1}\right)\right)\right| dT\, dS.$$

Proof. The derivative of the map (2.17) at (S, T) coincides with the following linear map:

$$(\Delta S, \Delta T) \to \left(\frac{1}{4}\Delta S \cdot T^{-1}S + \frac{1}{4}S \cdot T^{-1} \cdot \Delta S - \frac{1}{4}ST^{-1} \cdot \Delta T \cdot T^{-1}S + \Delta T + \Delta S,\right.$$
$$\left.\frac{1}{4}\Delta S \cdot T^{-1}S + \frac{1}{4}S \cdot T^{-1} \cdot \Delta S - \frac{1}{4}ST^{-1} \cdot \Delta T \cdot T^{-1}S + \Delta T - \Delta S\right).$$

After a linear transformation, the right-hand side becomes

$$\left(\Delta S, \frac{1}{4}\Delta S \cdot T^{-1}S + \frac{1}{4}S \cdot T^{-1} \cdot \Delta S - \frac{1}{4}ST^{-1} \cdot \Delta T \cdot T^{-1}S + \Delta T\right).$$

Hence, the determinant of the above is a constant multiple of the determinant of the following linear map:

$$\Delta T \to \Delta T - \frac{1}{4}ST^{-1} \cdot \Delta T \cdot T^{-1}S = \left(1 - \rho\left(\frac{1}{2}ST^{-1}\right)\right)\Delta T. \qquad \square$$

The following lemma is well known, but for completeness we include a simple proof.

LEMMA 2.20. *Let $r = 2 \dim_{\mathbf{R}} \mathscr{H}/\dim_{\mathbf{R}} V$ (notice here that* $\dim_{\mathbf{R}} \mathscr{H} = \dim_{\mathbf{R}} \mathfrak{g}$*). Then, for $d' \geqslant d$, there is* const > 0 *such that for a test function* ψ

$$\int_{M_{d,d'}(\mathbf{D})} \psi(x\bar{x}^t)\, dx = \text{const} \int_{\mathscr{H}^+} \psi(P)|\det_{\mathbf{R}} P|^{(d'-r)/2}\, dP.$$

Proof. Let $g \in GL(V)$. Then

$$\int_{M_{d,d'}(\mathbf{D})} \psi(gx\bar{x}^t\bar{g}^t)\,dx = |\det_{\mathbf{R}} g|^{-d'} \int_{M_{d,d'}(\mathbf{D})} \psi(x\bar{x}^t)\,dx$$

and

$$\begin{aligned}\int_{\mathscr{H}+} &\psi(gP\bar{g}^t)|\det_{\mathbf{R}} P|^{(d'-r)/2}\,dP \\ &= \int_{\mathscr{H}+} \psi(P)|\det_{\mathbf{R}} g^{-1}P(\bar{g}^{-1})^t|^{(d'-r)/2}|\det_{\mathbf{R}} g|^{-r}\,dP \\ &= |\det_{\mathbf{R}} g|^{-d'} \int_{\mathscr{H}+} \psi(P)|\det_{\mathbf{R}} P|^{(d'-r)/2}\,dP. \qquad \square\end{aligned}$$

LEMMA 2.21. *Let*

$$(\mathfrak{g} \times GL(V))^+ = \{(z, g) \in \mathfrak{g} \times GL(V);\, 4I_d > (\bar{g}^t)^{-1}Fzg^{-1}(\overline{(\bar{g}^t)^{-1}Fzg^{-1}})^t\}.$$

Set

$$\begin{aligned}M(z, g) = &\left|\det_{\mathbf{R}}\left(\frac{1}{4}(\bar{g}^t)^{-1}Fzg^{-1}(\overline{(\bar{g}^t)^{-1}Fzg^{-1}})^t + 1 - (\bar{g}^t)^{-1}Fzg^{-1}\right)\right|^{(p-r)/2} \\ &\times \left|\det_{\mathbf{R}}\left(\frac{1}{4}(\bar{g}^t)^{-1}Fzg^{-1}(\overline{(\bar{g}^t)^{-1}Fzg^{-1}})^t + 1 + (\bar{g}^t)^{-1}Fzg^{-1}\right)\right|^{(q-r)/2} \\ &\times \left|\det_{\mathbf{R}}\left(1 - \rho\left(\frac{1}{2}(\bar{g}^t)^{-1}Fzg^{-1}\right)\right)\right| |\det_{\mathbf{R}}(g)|^{d'-r}.\end{aligned}$$

One can normalize all the measures involved so that for a test function $\phi \in C_c(W^{\max})$

$$\int_W \phi(w)\,dw = \int_{(\mathfrak{g}\times GL(V))^+} \int_{L'} \phi(kg(\sigma_{\mathfrak{g}}(z)))\,dk\, M(z, g)\,dz\,dg.$$

Proof. Define a function ψ on $\mathfrak{l}$ by

$$\psi \circ \tau_{\mathfrak{l}}(w) = \int_{L'} \phi(wk)\,dk.$$

Since there is a matrix u such that $u\bar{u}^t = I_d$ and $uF'\bar{u}^t = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, (2.2) and

(2.20) imply that

$$\begin{aligned}\int_W \phi(w)\,dw &= \int_W \psi\circ\tau_{\mathrm{I}}(w)\,dw\\ &= \mathrm{const}\int_{M_{d,p}(\mathbf{D})}\int_{M_{d,q}(\mathbf{D})}\psi(x\bar{x}^tF,\ -y\bar{y}^tF)\,dy\,dx\\ &= \mathrm{const}\int_{\mathscr{H}^+}\int_{\mathscr{H}^+}\psi(PF,\ -P'F)|\det{}_{\mathbf{R}}P|^{(p-r)/2}|\det{}_{\mathbf{R}}P'|^{(q-r)/2}\,dP'\,dP.\end{aligned}$$

By (2.19), the above is a constant multiple of

$$\begin{aligned}\int_{(\mathscr{H}\times\mathscr{H}^+)^+}&\psi\left(\left(\frac{1}{4}ST^{-1}S+T+S\right)F,\ -\left(\frac{1}{4}ST^{-1}S+T-S\right)F\right)\\ &\times\left|\det{}_{\mathbf{R}}\left(\frac{1}{4}ST^{-1}S+T+S\right)\right|^{(p-r)/2}\\ &\times\left|\det{}_{\mathbf{R}}\left(\frac{1}{4}ST^{-1}S+T-S\right)\right|^{(q-r)/2}\\ &\times\left|\det{}_{\mathbf{R}}\left(1-\rho\left(\frac{1}{2}ST^{-1}\right)\right)\right|\,dT\,dS.\end{aligned}$$

Let us write $S = zF^{-1}$ and $T = (\overline{F^{-1}})^t\bar{g}^tgF^{-1}$, as in (2.8). Then again by (2.20), the above is a constant multiple of

$$\begin{aligned}\int_{(\mathfrak{g}\times GL(V))^+}&\psi(\tau_{\mathrm{I}}(g(\sigma_{\mathfrak{g}}(z))))\left|\det{}_{\mathbf{R}}\left(\frac{1}{4}zg^{-1}(\bar{g}^t)^{-1}\bar{z}^t+(\bar{F}^t)^{-1}\bar{g}^tgF^{-1}+zF^{-1}\right)\right|^{(p-r)/2}\\ &\times\left|\det{}_{\mathbf{R}}\left(\frac{1}{4}zg^{-1}(\bar{g}^t)^{-1}\bar{z}^t+(\bar{F}^t)^{-1}\bar{g}^tgF^{-1}-zF^{-1}\right)\right|^{(q-r)/2}\end{aligned}$$

$$\times\left|\det{}_{\mathbf{R}}\left(1-\rho\left(\frac{1}{2}zg^{-1}(\bar{g}^t)^{-1}\bar{F}^t\right)\right)\right||\det{}_{\mathbf{R}}g|^r\,dg\,dz. \tag{2.22}$$

Using the relation

$$\det{}_{\mathbf{R}}(A+(\bar{F}^t)^{-1}\bar{g}^tgF^{-1}) = \det{}_{\mathbf{R}}(\bar{F}^t(\bar{g}^t)^{-1}AFg^{-1}+1)|\det(g)|^2,$$

one can transform (2.22) to obtain the integral formula of (2.21). □

Finally, we make a specific choice of the matrix F:

$$F = \begin{cases} \begin{pmatrix} 0 & I_{d/2} \\ -I_{d/2} & 0 \end{pmatrix} & \text{if } \mathbf{D} = \mathbf{R} \\ \begin{pmatrix} iI_s & 0 \\ 0 & 0 - iI_{d-s} \end{pmatrix} & \text{if } \mathbf{D} = \mathbf{C} \\ iI_d & \text{if } \mathbf{D} = \mathbf{H}. \end{cases}$$

Also, let

$$B^+ = \begin{cases} \{b = \operatorname{diag}(b_1, \ldots, b_d);\, b_1 = b_{d/2+1} > b_2 \\ \qquad = b_{d/2+2} > \cdots > b_{d/2} = b_d > 0\} & \text{if } \mathbf{D} = \mathbf{R} \\ \{b = \operatorname{diag}(b_1, \ldots, b_d);\, b_1 > b_2 > \cdots > b_s > 0, \\ \qquad b_{s+1} > b_{s+2} > \cdots > b_d > 0\} & \text{if } \mathbf{D} = \mathbf{C} \\ \{b = \operatorname{diag}(b_1, \ldots, b_d);\, b_1 > b_2 > \cdots > b_d > 0\} & \text{if } \mathbf{D} = \mathbf{H}. \end{cases}$$

There is a function $\delta(b)$, $b \in B^+$, [S, 8.1.1] such that

$$\int_{GL(V)} f(g)\, dg = \int_U \int_{B^+} \int_G f(ubh)\delta(b)\, du\, db\, dh, \qquad \text{and}$$

$$\delta(b) \leqslant \text{const} \cdot (b_1^{d-1} b_2^{d-3} \cdots b_d^{-d+1})^n, \qquad n = \dim_{\mathbf{R}}(\mathbf{D}). \tag{2.23}$$

Finally, we arrive at a precise formulation of the Theorem 1.6 (a).

THEOREM 2.24. *Let* $(\mathfrak{g} \times B^+)^+ = \{(z, b) \in \mathfrak{g} \times B^+;\, 4I_d > (b^{-1}Fzb^{-1})(\overline{b^{-1}Fzb^{-1}})^t\}$. *Let*

$$\begin{aligned} m(z, b) = {} & \left|\det_{\mathbf{R}}\left(\frac{1}{4} b^{-1}Fzb^{-1}(\overline{b^{-1}Fzb^{-1}})^t + 1 - b^{-1}Fzb^{-1}\right)\right|^{(p-r)/2} \\ & \times \left|\det_{\mathbf{R}}\left(\frac{1}{4} b^{-1}Fzb^{-1}(\overline{b^{-1}Fzb^{-1}})^t + 1 + b^{-1}Fzb^{-1}\right)\right|^{(q-r)/2} \\ & \times \left|\det_{\mathbf{R}}\left(1 - \rho\left(\frac{1}{2} b^{-1}Fzb^{-1}\right)\right)\right| |\det_{\mathbf{R}} b|^{d'-r} \delta(b). \end{aligned}$$

Then, with appropriate normalization of all the measures involved,

$$\int_W \phi(w)\, dw = \int_G \int_{(\mathfrak{g} \times B^+)^+} \int_{L'} \phi(h \cdot \sigma_{\mathfrak{g}}(z) \cdot b^{-1}k^{-1})\, dk\, m(z, b)\, db\, dz\, dh.$$

Proof. We apply the equation (2.23) to (2.21) by writing $g = ubh$ and then changing z to $h^{-1}zh$. □

Proof of Theorem 1.6. It remains to show Theorem 1.6 (b) and the last statement. Let ψ be a continuous rapidly decreasing function on $\mathfrak{g}'$. Then

$$\mathscr{A}\psi(z) = \int_{(\mathfrak{g}\times B^+)^+} \int_{L'} \psi \circ \tau_{\mathfrak{g}'}(\sigma_{\mathfrak{g}}(z)b^{-1}k^{-1})\, dk\, m(z, b)\, db. \tag{2.25}$$

Since the function m is bounded on $(\mathfrak{g} \times B^+)^+$, it is clear from (2.7) that

$$\mathscr{A}\psi(z) \leqslant \text{const} \int_{B^+} \int_{L'} |\psi \circ \tau_{\mathfrak{g}'}(\sigma_{\mathfrak{g}}(z)b^{-1}k^{-1})|\, |\det_{\mathbf{R}} b|^{d'-r}\delta(b)\, dk\, db$$

$$\times \text{const}_N \int_{B^+} (1 + |bzb^{-1}|)^{-N}(1 + |bF^{-1}b|)^{-N}|\det_{\mathbf{R}} b|^{d'-r}\delta(b)\, db.$$

Notice that

$$\begin{aligned}|bzb^{-1}|^2 = |bFzb^{-1}|^2 &= \sum_{i,j} b_i^2|(Fz)_{i,j}|^2 b_j^{-2}\\ &= \sum_i |(Fz)_{i,i}|^2 + \sum_{i<j} |(Fz)_{i,j}|^2(b_i^2 b_j^{-2} + b_i^{-2}b_j^2)\\ &\geqslant \sum_i |(Fz)_{i,i}|^2 + 2\sum_{i<j} |(Fz)_{i,j}|^2 = |Fz|^2 = |z|^2.\end{aligned}$$

Further, the inequality (2.23) implies that for $N > 0$ large enough

$$\int_{B^+} (1 + |bF^{-1}b|)^{-N}|\det_{\mathbf{R}} b|^{d'-r}\delta(b)\, db < \infty. \tag{2.26}$$

Thus,

$$|\mathscr{A}\psi(z)| \leqslant \text{const}'_N(1 + |z|)^{-N} \qquad (z \in \mathfrak{g}).$$

This verifies Theorem 1.6 (b).

If $\psi \in C^\infty(\mathfrak{g}')$ and $\operatorname{supp} \psi \cap \tau_{\mathfrak{g}'}(W^{\max})$ is compact, then we integrate over a compact subset of $(\mathfrak{g} \times B^+)^+$ in (2.25). The projection of this set on B^+ is also compact. Thus, we may take derivatives with respect to $z \in \mathfrak{g}$ and estimate as above *without appealing to the inequality* (2.26). Hence, the last statement follows. □

3. The case when $(\, , \,)'$ is skew-symmetric and $\mathbf{D} = \mathbf{R}$ or $\mathbf{C}$. In this section, $\mathbf{D}$ is equipped with the trivial involution. Let $d' \geqslant 2d$ be positive integers with d' even. Let $V = \mathbf{D}^d$ and let $V' = \mathbf{D}^{d'}$. Fix a nonsingular matrix $F \in M_d(\mathbf{R})$ such that

$F = F^t = F^{-1}$. Let

$$F' = \begin{bmatrix} 0 & 0 & I_d \\ 0 & F'' & 0 \\ -I_d & 0 & 0 \end{bmatrix}, \qquad F'' = \begin{pmatrix} 0 & I_{d'/2-d} \\ -I_{d'/2-d} & 0 \end{pmatrix}. \tag{3.1}$$

Set

$$(u, v) = u^tFv, \quad (u', v')' = u'^tF'v' \qquad (u, v \in V, u', v' \in V'). \tag{3.2}$$

Then $(\,,\,)$ is a nondegenerate symmetric form on V, and $(\,,\,)'$ is a nondegenerate skew-symmetric form on V'. The corresponding isometry groups and Lie algebras can be represented in terms of matrices as follows:

$$G = \{g \in M_d(\mathbf{D}); g^tFg = F\}, \qquad \mathfrak{g} = \{z \in M_d(\mathbf{D}); z^tF + Fz = 0\},$$

$$G' = \{g \in M_{d'}(\mathbf{D}); g^tF'g = F'\}, \qquad \mathfrak{g}' = \{z \in M_{d'}(\mathbf{D}); z^tF' + F'z = 0\}.$$

Let $L' = \{g \in G'; \bar{g}^tg = I_{d'}\}$, where $g \to \bar{g}$ is the complex conjugation if $\mathbf{D} = \mathbf{C}$, and is trivial otherwise. This is a maximal compact subgroup of G'. Let us view the quaternions as matrices

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad a, b \in \mathbf{C}.$$

Let $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We identify

$$\mathbf{C} \ni a \to \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \in \mathbf{H}.$$

Then

$$\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j, \qquad aj = j\bar{a}, a \in \mathbf{C}. \tag{3.3}$$

Let

$$\mathbf{D}' = \begin{cases} \mathbf{C} & \text{if } \mathbf{D} = \mathbf{R} \\ \mathbf{H} & \text{if } \mathbf{D} = \mathbf{C}. \end{cases}$$

If $\mathbf{D}' = \mathbf{H}$, let $j \in \mathbf{D}'$ be as in (3.3). If $\mathbf{D}' = \mathbf{C}$, let $j \in \mathbf{D}'$ be $\sqrt{-1}$. Then

$$\mathbf{D}' = \mathbf{D} \oplus \mathbf{D}j. \tag{3.4}$$

Let $\mathbf{D}' \ni a \to \bar{a} \in \mathbf{D}'$ denote the standard nontrivial involution over $\mathbf{R}$. Let

$$L = \{g \in M_d(\mathbf{D}'); \bar{g}^t jFg = jF\}, \qquad \text{and}$$

$$\mathfrak{l} = \{x \in M_d(\mathbf{D}'); \bar{x}^t jF + jFx = 0\}.$$

Notice that if $G \cong O_{p,q}$, then $L \cong U_{p,q}$, and if $G \cong O_p(\mathbf{C})$, then $L \cong O^*_{2p}$.

Let $W = M_{d,d'}(\mathbf{D})$. The groups G, G' act on W as indicated in the introduction. The moment maps $\tau_{\mathfrak{g}}$: $W \to \mathfrak{g}$, $\tau_{\mathfrak{l}}$: $W \to \mathfrak{l}$ and $\tau_{\mathfrak{g}'}$: $W \to \mathfrak{g}'$ are given explicitly by

$$\text{(3.5)} \quad \tau_{\mathfrak{g}}(w) = wF'w^tF, \quad \tau_{\mathfrak{l}}(w) = (wF'w^t + w\bar{w}^tj)F, \quad \tau_{\mathfrak{g}'}(w) = F'w^tFw \qquad (w \in W).$$

As in (2.3), we have a decomposition

$$\text{(3.6)} \qquad W = M_d(\mathbf{D}) \oplus M_{d,d'-2d}(\mathbf{D}) \oplus M_d(\mathbf{D})$$

in terms of which we define a section of the map τ:

$$\text{(3.7)} \qquad \sigma_{\mathfrak{g}}(z) = \left(\frac{1}{2}z, 0, F^{-1}\right) \qquad (z \in \mathfrak{g}).$$

We identify $GL(V)$ with a subgroup of G' by a formula analogous to (2.5):

$$GL(V) \ni g \to \begin{bmatrix} g & 0 & 0 \\ 0 & I_{d'-2d} & 0 \\ 0 & 0 & (g^t)^{-1} \end{bmatrix} \in G'.$$

Then, for $g \in GL(V)$ and $z \in \mathfrak{g}$,

$$\text{(3.8)} \qquad g(\sigma_{\mathfrak{g}}(z)) = \sigma_{\mathfrak{g}}(z)\cdot g^{-1} = \left(\frac{1}{2}zg^{-1}, 0, F^{-1}g^t\right).$$

With the notation of (3.8), we have

$$\tau_{\mathfrak{g}}(g(\sigma_{\mathfrak{g}}(z))) = z$$

$$\tau_{\mathfrak{l}}(g(\sigma_{\mathfrak{g}}(z))) = \left(\frac{1}{4}zg^{-1}(\bar{g}^t)^{-1}\bar{z}^t + Fg^t\bar{g}F - zjF\right)jF$$

$$\text{(3.9)} \qquad \tau_{\mathfrak{g}'}(g(\sigma_{\mathfrak{g}}(z))) = \begin{bmatrix} \frac{1}{2}gzg^{-1} & 0 & gFg^t \\ 0 & 0 & 0 \\ -\frac{1}{4}(g^t)^{-1}z^tFzg^{-1} & 0 & -\frac{1}{2}(g^t)^{-1}z^tg^t \end{bmatrix}.$$

With the notation of (3.9), let $S = zF$ and let $T = Fg^t\bar{g}F$. Then,

$$\tau_1(g(\sigma_g(z))) = \left(\frac{1}{4}(Sj)T^{-1}(\overline{Sj})^t + T - Sj\right)jF = \left(\frac{1}{4}S(\overline{T})^{-1}\overline{S}^t + T - Sj\right)jF,$$

(3.10)

$$S = -S^t, \qquad T = \overline{T}^t, T > 0.$$

Let $\mathscr{S} = \{S \in M_d(\mathbf{D}); S = -S^t\}$ be the space of skew-symmetric matrices of size d. Let $\mathscr{H}^+ = \mathscr{H}^+(\mathbf{D})$ be the set of positive hermitian matrices of size d as before. Similarly, we define $\mathscr{H}^+(\mathbf{D}')$.

LEMMA 3.11. *Let* $(\mathscr{S} \times \mathscr{H}^+)^+ = \{(S, T) \in \mathscr{S} \times \mathscr{H}^+;\ T > (1/4)S\overline{T}^{-1}\overline{S}^t\}$. *Then the map*

$$(\mathscr{S} \times \mathscr{H}^+)^+ \ni (S, T) \to \frac{1}{4}S\overline{T}^{-1}\overline{S}^t + T - Sj \in \mathscr{H}(\mathbf{D}')^+$$

is a bijection.

Proof. The group $GL(V) = GL(\mathbf{D}^d) \subseteq GL(\mathbf{D}'^d)$ acts on $\mathscr{S}$, $\mathscr{H}(\mathbf{D})$, $\mathscr{H}(\mathbf{D}')$ by

$$g(S) = gSg^t,\ g(T) = gT\bar{g}^t,\ g(P) = gP\bar{g}^t \qquad (g \in GL(V), S \in \mathscr{S}, T \in \mathscr{H}, P \in \mathscr{H}(\mathbf{D}')).$$

Moreover, we have the following formula:

$$g\left(\frac{1}{4}S\overline{T}^{-1}\overline{S}^t + T - Sj\right)\bar{g}^t = \frac{1}{4}(gSg^t)(\overline{gT\bar{g}^t})^{-1}(\overline{gSg^t})^t + (gT\bar{g}^t) - (gSg^t)j.$$

Clearly, the action of $GL(V)$ on $\mathscr{S} \times \mathscr{H}^+$ preserves $(\mathscr{S} \times \mathscr{H}^+)^+$. Fix $S \in \mathscr{S}$. Given $P \in \mathscr{H}^+(\mathbf{D}')$ such that $P - Sj \in \mathscr{H}^+(\mathbf{D}')$, we will show that there is a unique $T \in \mathscr{H}^+(\mathbf{D})$ such that

$$\frac{1}{4}S\overline{T}^{-1}\overline{S}^t + T - Sj = P. \tag{3.12}$$

Using the action of $GL(V)$, we may assume that

$$\frac{1}{2}S = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The stabilizer of $(1/2)S$ in $GL(V)$ consists of matrices of the form

$$g = \begin{pmatrix} h & B \\ 0 & C \end{pmatrix}, \qquad hEh^t = E, \det(C) \neq 0.$$

As in the proof of (2.9), we may reduce to the case $(1/2)S = E$ and complete the proof. □

For $g \in GL(V)$ and $T \in \mathscr{H}$, set

$$\rho(g)T = g\overline{T}g^t.$$

Then $\rho(g) \in \mathrm{End}_{\mathbf{R}}(\mathscr{H})$. As in (2.19), one can verify the following lemma.

LEMMA 3.13. *One can normalize the Lebesgue measures dQ, dT, dS on $\mathscr{H}(\mathbf{D}')$, $\mathscr{S}$, $\mathscr{H}(\mathbf{D})$, respectively, so that for a test function ψ*

$$\int_{\mathscr{H}^+(\mathbf{D}')} \psi(Q)\, dQ = \int_{(\mathscr{S}\times\mathscr{H}^+)^+} \psi\left(\frac{1}{4}S(\overline{T})^{-1}(\overline{S})^t + T - Sj\right) \times \left|\det_{\mathbf{R}}\left(1 - \rho\left(\frac{1}{2}S(\overline{T})^{-1}\right)\right)\right| dT\, dS.$$

LEMMA 3.14. *Let*

$$(\mathfrak{g} \times GL(V))^+ = \{(z, g) \in \mathfrak{g} \times GL(V);\, 4I_d > ((g^t)^{-1}Fzg^{-1})(\overline{(g^t)^{-1}Fzg^{-1}})^t\}.$$

Let $r' = 2\dim_{\mathbf{R}} \mathscr{H}(\mathbf{D}')/\dim_{\mathbf{R}}(\mathbf{D}'^d)$ and let $r = 2\dim_{\mathbf{R}} \mathscr{H}(\mathbf{D})/\dim_{\mathbf{R}}(\mathbf{D}^d)$. Set

$$M(z, g) = \left|\det_{\mathbf{R}}\left(\frac{1}{4}(g^t)^{-1}Fzg^{-1}(\overline{(g^t)^{-1}Fzg^{-1}})^t + 1 - (g^t)^{-1}Fzg^{-1}Fj\right)\right|^{d'/2-r'} \times \left|\det_{\mathbf{R}}\left(1 - \rho\left(\frac{1}{2}(g^t)^{-1}Fzg^{-1}\right)\right)\right| |\det_{\mathbf{R}}(g)|^{d'-2r'+r}.$$

One can normalize all the measures involved so that for a test function $\phi \in C_c(W^{\max})$,

$$\int_W \phi(w)\, dw = \int_{(\mathfrak{g}\times GL(V))^+} \int_{L'} \phi(kg(\sigma_{\mathfrak{g}}(z)))\, dk\, M(z, g)\, dz\, dg.$$

Proof. Define a function ψ on $\mathfrak{l}$ by

$$\psi \in \tau_{\mathfrak{l}}(w) = \int_{L'} \phi(wk)\, dk.$$

Then by (3.5) and (3.13),

$$\int_W \phi(w)\, dw = \int_W \psi \circ \tau_{\mathfrak{l}}(w)\, dw$$

$$\int_W \psi((-wF'w'j + w\overline{w}^t)jF)\, dw = \int_W \psi(w(1 - F'j)\overline{w}^t jF)\, dw.$$

Set $D = \begin{pmatrix} 0 & I_{d'/2-d} \\ I_d & 0 \end{pmatrix}$ so that $F' = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$; see (3.1). There is an isomorphism of vector spaces

$$M_{d,d'}(\mathbf{D}) = M_{d,d'/2}(\mathbf{D}) \oplus M_{d,d'/2}(\mathbf{D}) \ni w = (A, B) \to A + BDj \in M_{d,d'/2}(\mathbf{D}'),$$

and the formula

$$w(1 - F'j)\overline{w}^t = (A + BDj)\overline{(A + BDj)}^t.$$

Thus, by Lemmas 2.20 and 3.13:

$$\begin{aligned} \int_W \phi(w)\,dw &= \int_{M_{d,d'/2}(\mathbf{D}')} \psi(x\overline{x}^t jF)\,dx \\ &= \int_{\mathscr{H}^+(\mathbf{D}')} \psi(PjF)|\det_{\mathbf{R}} P|^{(d'/2-r')/2}\,dP \\ &= \int_{(\mathscr{S}\times\mathscr{H}^+)^+} \psi\left(\left(\frac{1}{4}S(\overline{T})^{-1}\overline{S}^t + T - Sj\right)jF\right) \\ &\quad \times \left|\det_{\mathbf{R}}\left(\frac{1}{4}S(\overline{T})^{-1}\overline{S}^t + T - Sj\right)\right|^{(d'/2-r')/2} \\ &\quad \times \left|\det_{\mathbf{R}}\left(1 - \rho\left(\frac{1}{2}S(\overline{T})^{-1}\right)\right)\right|\,dT\,dS. \end{aligned}$$

If we write $S = zF$ and $T = Fg^t\overline{g}F$ as in (3.10), then, again by Lemma 2.20,

$$\begin{aligned} \int_W \phi(w)\,dw = \int_{(\mathfrak{g}\times GL(V))^+} &\psi\left(\left(\frac{1}{4}zg^{-1}(\overline{g}^t)^{-1}\overline{z}^t + Fg^t\overline{g}F - zFj\right)jF\right) \\ &\times \left|\det_{\mathbf{R}}\left(\frac{1}{4}zg^{-1}(\overline{g}^t)^{-1}\overline{z}^t + Fg^t\overline{g}F - zFj\right)\right|^{(d'/2-r')/2} \\ &\times \left|\det_{\mathbf{R}}\left(1 - \rho\left(\frac{1}{2}zg^{-1}(\overline{g}^t)^{-1}F\right)\right)\right| |\det_{\mathbf{R}} g|^r\,dg\,dz, \end{aligned}$$

and the lemma follows. □

Let us make a specific choice

$$F = \begin{cases} \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} & \text{if } \mathbf{D} = \mathbf{R} \\ \qquad I_d & \text{if } \mathbf{D} = \mathbf{C}. \end{cases}$$

Let

$$B^+ = \begin{cases} \{b = \operatorname{diag}(b_1, \ldots, b_d);\, b_1 > \cdots > b_p > 0,\, b_{p+1} > \cdots > b_d > 0\} & \text{if } \mathbf{D} = \mathbf{R} \\ \{b = \operatorname{diag}(b_1, \ldots, b_d);\, b_1 > \cdots > b_d > 0\} & \text{if } \mathbf{D} = \mathbf{C}. \end{cases}$$

There is a function $\delta(b)$, $b \in B^+$, [S, 8.1.1] such that

$$\int_{GL(V)} f(g)\, dg = \int_U \int_{B^+} \int_G f(ubh)\delta(b)\, du\, db\, dh, \qquad \text{and}$$

$$\delta(b) \leqslant \text{const} \cdot (b_1^{d-1} b_2^{d-3} \cdots b_d^{-d+1})^n, \qquad n = \dim_{\mathbf{R}}(\mathbf{D}).$$

Finally, we arrive at a precise formulation of Theorem 1.6 (a), which can be verified the same way as Theorem 2.24.

THEOREM 3.15. *Let* $(\mathfrak{g} \times B^+)^+ = \{(z, b) \in \mathfrak{g} \times B^+;\, 4I_d > (b^{-1}Fzb^{-1})(\overline{b^{-1}Fzb^{-1}})^t\}$. *Let*

$$m(z, b) = \left|\det_{\mathbf{R}}\left(\frac{1}{4} b^{-1}Fzb^{-1}(\overline{b^{-1}Fzb^{-1}})^t + 1 - b^{-1}Fzb^{-1}Fj\right)\right|^{(d'/2 - r')/2}$$

$$\times \left|\det_{\mathbf{R}}\left(1 - \rho\left(\frac{1}{2} b^{-1}Fzb^{-1}\right)\right)\right| |\det_{\mathbf{R}} b|^{d' - 2r' + r} \delta(b).$$

Then,

$$\int_W \phi(w)\, dw = \int_G \int_{(\mathfrak{g} \times B^+)^+} \int_{L'} \phi(h \cdot \sigma_{\mathfrak{g}}(z) \cdot b^{-1}k^{-1})\, dk\, m(z, b)\, db\, dz\, dh.$$

As before,

$$\mathscr{A}\psi(z) = \int_{(\mathfrak{g} \times B^+)^+} \int_{L'} \psi \circ \tau_{\mathfrak{g}'}(\sigma_{\mathfrak{g}}(z) b^{-1}k^{-1})\, dk\, m(z, b)\, db. \tag{3.16}$$

Hence, the proof of the remaining statements of Theorem 1.6 is the same as in the previous case.

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